Math 255A' Lecture 17 Notes

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1 Projections and Idempotents in Hilbert Spaces

1.1 Projections and idempotents

Let H be a Hilbert space over $\mathbb{F}$.

Definition 1.1. An operator $E \in \mathcal{B}(H)$ is **idempotent** if $E^2 = E$. E is a **projection** if $E^2 = E$ and $\ker E = (\operatorname{ran} E)^\perp$.

Proposition 1.1. *Let $E \in \mathcal{H}$.*

1. *E is idempotent if and only if $1 - E$ is idempotent.*
2. *$\operatorname{ran} E = \ker(1-E)$, $\ker E = \operatorname{ran}(1-E)$, and these are closed subspaces of H.*
3. *$\ker E \cap \operatorname{ran} E = \{0\}$, and $\ker E + \operatorname{ran} E = H$.*

Proof. 1. $(1-E)^2 = 1 - 2E + E^2$.

2.

$$\begin{aligned} h \in \operatorname{ran} E &\iff hEk \text{ for some } k \\ &\iff Eh = E^2 k = Ek = h \\ &\iff (1-E)h = 0. \end{aligned}$$

3. $h = Eh + (1-E)h$. □

Remark 1.1. This also holds for Banach spaces, but we will not use it in that generality.

Proposition 1.2. *Let P be a nonzero idempotent in $\mathcal{B}(H)$. The following are equivalent:*

1. *P is a projection.*
2. *P is the projection onto $\operatorname{ran} P$.*

3. $\|P\| = 1$.

4. $P = P^*$.

5. P *is normal.*

6. $\langle Ph, h \rangle \geq 0$ *for all* h *(nonnegativity).*

Proof. (1) $\implies$ (2): Let $M = \operatorname{ran} P$, which is closed. Then the projection $P_M h$ is characterized by $P_M h - h \perp M$; we show that P has this property. For any $h \in H$, $h - Ph = (1 - P)h \in \operatorname{ran}(1 - P) = \ker P$, and $\operatorname{ran} P \subseteq (\ker P)^\perp$. So $h - Ph \perp M$.

(2) $\implies$ (3): Write $h_1 = Ph$, so $h = h_1 + (h - h_1)$. Then $\|h_1\| \leq \|h\|$ if $h \in M$.

(3) $\implies$ (1): We want to show that $\ker P = (\operatorname{ran} P)^\perp$. We will show that $(\ker P)^\perp = \operatorname{ran} P$. Assume $h \perp \ker P$; we will deduce that $h \in \operatorname{ran} P$. We get

$$0 = \langle h, h - Ph \rangle \implies \|h\|^2 = \langle h, Ph \rangle \implies \|h\|^2 \leq \|h\| \|Ph\| \leq \|P\| \|h\|^2 = \|h\|^2,$$

so all these are equal. Then

$$\|h - Ph\|^2 = \|h\|^2 + \|Ph\|^2 - 2 \operatorname{Re} \langle h, Ph \rangle = 0,$$

so $h \in \operatorname{ran} P$.

Suppose $h \in \operatorname{ran} P$. Then $h = h_1 = h_2$, where $h_1 \in (\ker P)^\perp$, and $h_2 \in \operatorname{ran} P$. and is orthogonal to ranP $\cap (\ker P)^\perp$. This means $h_2 \in \operatorname{ran} P \cap \ker P = \{0\}$. so $h = h_1$.

(2) $\implies$ (4): Suppose $P = P_M$. Then $h = h_1 + h_2$ and $k = k_1 + k_2$, where $h_1, k_1 \in M$ and $h_2, k_2 \perp M$. Then

$$\langle Ph, k \rangle = \langle h_1, k_1 + k_2 \rangle = \langle h_1, k_1 \rangle = \langle h, Pk \rangle .$$

(4) $\implies$ (5): if $P = P^*$, then P commutes with P^*.

(5) $\implies$ (1): If $PP^* = P^*P$, then $\ker PP^* = \ker P^*P$. If $PP^*h = 0$, then

$$\langle PP^*h, h \rangle = \langle P^*h, P^*h \rangle = \|P^*h\|^2,$$

so multiplying by an adjoint does not change the kernel. So $\ker(PP^*) = \ker(P^*) = (\operatorname{ran} P)^\perp$. On the other hand, the same argument gives $\ker P^*P = \ker P$.

(6) $\implies$ (1): Suppose (1) does not hold, so there are an $h = Ph$ and $k \in \ker P$ such that $\langle h, k \rangle \neq 0$. Then

$$\langle P(\alpha h + \beta k), \alpha h + \beta k \rangle = \langle \alpha h, \alpha h + \beta k \rangle = \|\alpha h\|^2 + \alpha \overline{\beta} \langle h, k \rangle ,$$

where $\langle h, k \rangle$ is not necessarily ≥ 0. □

1.2 Invariant and reducible subspaces

If P is a projection, then the map $h \mapsto (Ph, h - P - h)$ is a Hilbert space isomorphism $H \to \operatorname{ran} P \oplus \ker P$. So if we have an operator on H, we can think of it as an operator acting on this direct sum. More generally, if we have a closed subspace M, then $H \cong M \oplus M^\perp$. If $A \in \mathcal{B}(H)$, we identify it with

$$\begin{bmatrix} X & Y \\ W & Z \end{bmatrix}, \qquad Ah = \begin{bmatrix} X & Y \\ W & Z \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix},$$

where $X \in \mathcal{B}(M)$, $Z \in \mathcal{B}(M^\perp)$, $Y \in \mathcal{B}(M^\perp, M)$, and $W \in \mathcal{B}(M, M^\perp)$. This gets us partway to diagonalization if we can show that $W, Y = 0$.

Definition 1.2. A subspace $M \leq H$ is **invariant** for $A \in \mathcal{B}(H)$ if $AM \subseteq M$. $M \leq H$ is **reducing** for $A \in \mathcal{B}(H)$ if $AM \subseteq M$ and $AM^\perp \subseteq M^\perp$.

Here's how we find X, Y, Z, W:

$$A(h_1+h_2) = PA(h_1+h_2)+(1-P)A(h_1+h_2) = PAPh+PA(1-P)h+(1-P)APh+(1-P)A(1-P)h.$$

In other words,

$$\begin{bmatrix} X & Y \\ W & Z \end{bmatrix} = \begin{bmatrix} PA|_M & PA|_{M^\perp} \\ (1-P)A|_M & (1-P)A|_{M^\perp} \end{bmatrix}.$$

Proposition 1.3. *1. M is invariant for A $\iff$ $PAP = AP$ $\iff$ $W = 0$.*

2. M is reducing for A $\iff$ $PA = AP$ $\iff$ $W = 0, Y = 0$.

Proof. 1. If $PAP = AP$, then $W = (1 - PA|_M = (1-P)AP|_M = 0$.

If M is invariant, then

$$\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} h_1 \\ 0 \end{bmatrix} = \begin{bmatrix} Xh_1 \\ 0 \end{bmatrix}.$$

2. If $PA = AP$, then $PAP = AP$, so M is invariant. On the other hand $PA = PAP$, so $PA(1-P) = 0$. So $M^\perp$ is invariant, making M reducing. □

Idea on the route to the spectral theorem for self-adjoint operators: Break A into

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$$

such that X, Y are both "simpler" than A was originally. Keep doing this to "diagonalize" A.